2-adic Valuation of Fibonacci-like Sequences

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Abstract

We examine the 2-adic valuation sequences associated with two different Fibonacci-like sequences. Then utilizing eigenvalues and eigenvectors of the associated matrices, we create closed formulas for these Fibonacci-like sequences. A valuation sequence is then created which is the focus of this research.

Introduction

A 2-adic valuation involves taking an integer's prime factorization. The exponent associated with the power of two is the integer's valuation.

$$\text{If } n = 2^k d, \gcd(2, d) = 1$$
$$\text{then } \nu_2(n) = k$$

Example: 2-adic valuation of 18 is 1 as the prime factorization is $2^1 * 3^2$ the exponent associated with 2 is 1.

Example: 2-adic valuation of 5 is 0 because no factors of 2 are present in the prime factorization.

Fibonacci-like sequences can be written in a simple recursive form where you add terms according to some rule.

Example: $F(n) = Prime(n) + Prime(n-1), Prime(0) = 2$

This recursive formula is the most intuitive way to write a Fibonacci-like sequence, however you wouldn't be able to calculate the nth variable. Recursive formulas are easily transformed into matrix formulas, then these matrix formulas can be used to calculate closed formulas for Fibonacci-like sequences. The closed formulas are the basis of near all other calculations.

Investigation Phase

For the first half of the project, we were focused on the background information and understanding. This resulted in most of the code and Excel files that were later utilized to find valuations sequences for the Fibonacci-like sequences.

To find the closed-form equations of Fibonacci-like sequences, the methodology used was taking some matrix form of the recursive equation and calculating the closed form equation. The process used to find the cases of the valuation sequences was to manually enter various starting values into the closed form Fibonacci-like formula and observe the returning valuation sequence. These closed formulas gave valuable insight to the underlying behavior in the sequence itself.

prime(n)		2^0	2^1	2^2	2^3	2^4	2^5	2^6	2^7	2^8	2^9	2^10	2^11	2^12	2^13	2^14	2^15	2^16	2^17	2^18	2^19	2^20
2		1	2	4	8	16	32	64	128	256	512	1024	2048	4096	8192	16384	32768	65536	1E+05	3E+05	5E+05	1E+06
3	5	0	1	1	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
5	8	0	0	0	0	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8
7	12	0	0	0	4	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12
11	18	0	0	2	2	2	18	18	18	18	18	18	18	18	18	18	18	18	18	18	18	18
13	24	0	0	0	0	8	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24
17	30	0	0	2	6	14	30	30	30	30	30	30	30	30	30	30	30	30	30	30	30	30
19	36	0	0	0	4	4	4	36	36	36	36	36	36	36	36	36	36	36	36	36	36	36
23	42	0	0	2	2	10	10	42	42	42	42	42	42	42	42	42	42	42	42	42	42	42
29	52	0	0	0	4	4	20	52	52	52	52	52	52	52	52	52	52	52	52	52	52	52
31	60	0	0	0	4	12	28	60	60	60	60	60	60	60	60	60	60	60	60	60	60	60
37	68	0	0	0	4	4	4	4	68	68	68	68	68	68	68	68	68	68	68	68	68	68
41	78	0	0	2	6	14	14	14	78	78	78	78	78	78	78	78	78	78	78	78	78	78
43	84	0	0	0	4	4	20	20	84	84	84	84	84	84	84	84	84	84	84	84	84	84
47	90	0	0	2	2	10	26	26	90	90	90	90	90	90	90	90	90	90	90	90	90	90
53	100	0	0	0	4	4	4	36	100	100	100	100	100	100	100	100	100	100	100	100	100	100
59	112	0	0	0	0	0	16	48	112	112	112	112	112	112	112	112	112	112	112	112	112	112
61	120	0	0	0	0	8	24	56	120	120	120	120	120	120	120	120	120	120	120	120	120	120
67	128	0	0	0	0	0	0	0	0	128	128	128	128	128	128	128	128	128	128	128	128	128
71	138	0	0	2	2	10	10	10	10	138	138	138	138	138	138	138	138	138	138	138	138	138
73	144	0	0	0	0	0	16	16	16	144	144	144	144	144	144	144	144	144	144	144	144	144
79	152	0	0	0	0	8	24	24	24	152	152	152	152	152	152	152	152	152	152	152	152	152
83	162	0	0	2	2	2	2	34	34	162	162	162	162	162	162	162	162	162	162	162	162	162
89	172	0	0	0	4	12	12	44	44	172	172	172	172	172	172	172	172	172	172	172	172	172
97	186	0	0	2	2	10	26	58	58	186	186	186	186	186	186	186	186	186	186	186	186	186

Figure 1. Excel graph showing prime Fibonacci-like valuation sequence

	2^0	2^1	2^2	2^3	2^4	2^5	2^6	2^7	2^8	2^9	2^10	2^11	2^12	2^13	2^14	2^15	2^16	2^17	2^18	2^19	2^20
2n+n	1	2	4	8	16	32	64	128	256	512	1024	2048	4096	8192	16384	32768	65536	1E+05	3E+05	5E+05	1E+06
3	0	1	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
6	0	0	2	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6
12	0	0	0	4	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12
24	0	0	0	0	8	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24
48	0	0	0	0	0	16	48	48	48	48	48	48	48	48	48	48	48	48	48	48	48
96	0	0	0	0	0	0	32	96	96	96	96	96	96	96	96	96	96	96	96	96	96
192	0	0	0	0	0	0	0	64	192	192	192	192	192	192	192	192	192	192	192	192	192
384	0	0	0	0	0	0	0	0	128	384	384	384	384	384	384	384	384	384	384	384	384
768	0	0	0	0	0	0	0	0	0	256	768	768	768	768	768	768	768	768	768	768	768
1536	0	0	0	0	0	0	0	0	0	0	512	1536	1536	1536	1536	1536	1536	1536	1536	1536	1536
3072	0	0	0	0	0	0	0	0	0	0	0	1024	3072	3072	3072	3072	3072	3072	3072	3072	3072
6144	0	0	0	0	0	0	0	0	0	0	0	0	2048	6144	6144	6144	6144	6144	6144	6144	6144
12288	0	0	0	0	0	0	0	0	0	0	0	0	0	4096	12288	12288	12288	12288	12288	12288	12288
24576	0	0	0	0	0	0	0	0	0	0	0	0	0	0	8192	24576	24576	24576	24576	24576	24576
49152	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	16384	49152	49152	49152	49152	49152
98304	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	32768	98304	98304	98304	98304
196608	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	65536	2E+05	2E+05	196608
393216	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1E+05	4E+05	393216
786432	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3E+05	786432
1572864	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	524288
3145728	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Figure 2. Excel graph showing doubled Fibonacci-like valuation sequence with $F_1 = 3$ and $F_2 = 6$

Results

Prime Fibonacci-like Sequence:

$$F(n) = Prime(n) + Prime(n-1), Prime(n_0 - 1) = 2$$

This formula will generate an unbounded sequence. There are an infinite amount of prime numbers and as such this sequence can't be bounded.

The valuation sequence of the Fibonacci-like sequence is unbound and definable. So that for any $n \in \mathbb{N}$:

$$F(n) = 2^{\ell} k + 2^{\ell-1}, \text{ where k is a non-negative integer}$$

Then the valuation will be:

$$\nu_n(F(n)) = \ell - 1, \ell \in \mathbb{N}$$

Doubled Fibonacci-like Sequence:

$$F_{n+1} = F_n + 2F_{n-1}$$

With the closed form:

$$F_n = \frac{1}{3}[F_2(2^n + (-1)^{n+1}) + F_1(2^n + 2(-1)^n)]$$

The doubled Fibonacci-like sequence is an unbounded sequence with the valuations depending on F_2 and the relationship between F_1 and F_2.

Case 1: F_2 is an odd number. The valuation sequence will be $\nu_2(F_1), 0, 0, 0, \ldots$

$$F_n = \frac{1}{3}[(-1)^n(2(j-k)-1) + 2^n(2k+j+1)], F_1 = j, F_2 = 2k+1$$

Case 2: $F_2 = 2F_1$ The valuation sequence is defined as

$$\nu_2(F_n) = n + \nu_2(k)$$
$$F_n = 2^n k, F_2 = 2k$$

Case 3: $F_2 \neq 2F_1 and\ F_2 = 2k, k \in \mathbb{N}$ The valuation sequence is bounded, but there is no general definition for n. The individual valuation sequences can be defined.

$$F_n = (-1)^n\left(\frac{2}{3}j - \frac{1}{3}m\right) + 2^n\left(\frac{1}{3}m + \frac{1}{3}j\right), F_1 = j, F_2 = m$$

Future Work

Additional research into the third case of the doubled Fibonacci-like sequence to find some pattern to the associated 2-adic valuation sequences.

Further research into a general Fibonacci-like sequence where both the n term and n-1 term are multiplied by some variable.

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