3-adic Valuations of $x^2 + a$

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Abstract

We seek to characterize the 3-adic valuations of the family of functions $x^2 + a$, and what values of a in one of three forms ($3\alpha + 1, 3\beta + 2, 3\gamma + 0$) determines $v_3(x^2 + a)$ to be. We partition the family of functions into three conjectures, depending on the three forms of a. Finally, we prove one of the three conjectures, that $v_3(x^2 + 3\alpha + 1) = 0$ for any natural number x.

Definitions

A 3-adic valuation of a number is a measure of how many factors of 3 are in a number's prime factorization. For example, the 3-adic valuation of 18 is 2 because there are 2 factors of 3 in the prime factorization of 18 ($18 = 3 * 3 * 2$). However, the number 7 is a different example: having no factors of 3 in its prime factorization, its 3-adic valuation is 0. More precisely, if $n = 3^k d$, where $gcd(3, d) = 1$, then $v_3(n) = k$.

A 3-adic valuation of a function's output can vary depending on the input of the function. To study the valuation of a function's output, we create a sequence of outputs $f(1), f(2), f(3)$, etc. that we can individually valuate. This gives us a sequence of 3-adic valuations to observe and discover patterns in. We can build trees that show the valuation of the function's output for each input, as shown in the figure on the right.

Conjectures

In valuating the family of functions $x^2 + a$, I have hypothesized that the 3-adic valuation of any function's output will consistently behave in one of three ways, determined by the form of a:

Conjecture Alpha (proven): If a is of the form $3\alpha + 1$, then the 3-adic valuation of $x^2 + a$ is 0 for all natural numbers x.

Conjecture Beta: If a is of the form $3\beta + 2$, then the 3-adic valuation of $x^2 + a$ is unbounded.

Conjecture Gamma: If a is of the form $3\gamma + 0$, then the 3-adic valuation of $x^2 + a$ is 1 for natural numbers x of the form 3t, and 0 for all other natural numbers x.

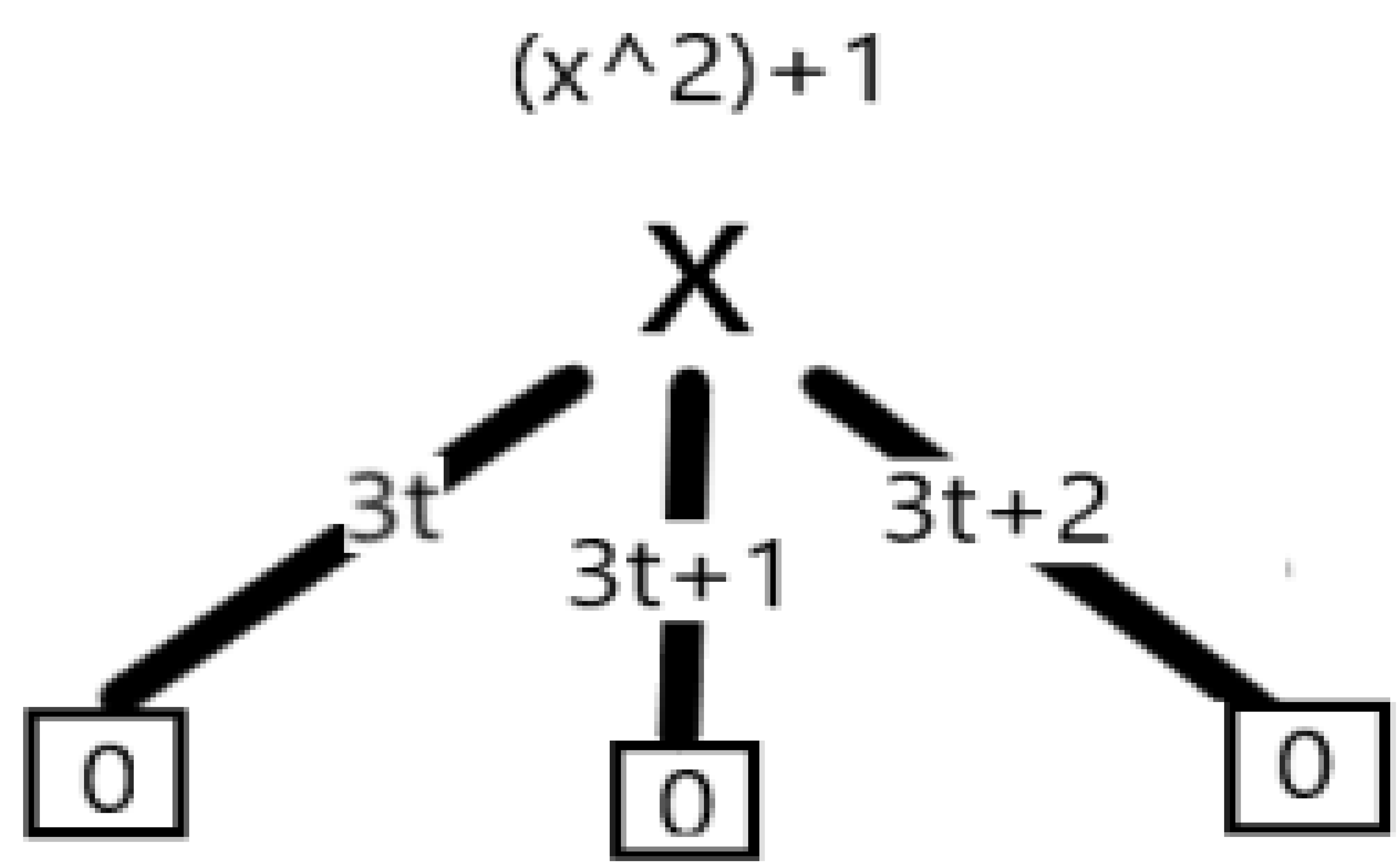

Figure 1. 3-adic Valuation Tree of x^2+1

Results and Conclusions

We found that 3-adic valuations of the family of functions $x^2 + a$ depends on the form of a. Furthermore, we can prove that if a is of the form $3\alpha + 1$, then $v_3(f(x)) = 0$ for all natural numbers x. Any natural number x can be written in exactly one of these three forms: $3t$, $3t + 1$, or $3t + 2$. When each of these forms are input into $x^2 + 3\alpha + 1$ for x, we obtain the following results:

$$9t^2 + 3\alpha + 1$$
$$9t^2 + 6t + 3\alpha + 1$$
$$9t^2 + 12t + 3\alpha + 5$$

Which can then be shown as:

$$3(3t^2 + \alpha) + 1$$
$$3(3t^2 + 2t + \alpha) + 1$$
$$3(3t^2 + 4t + \alpha) + 5$$

Because all three of these results are $3 * (something) + 1$ or $3 * (something) + 5$, none of these expressions are divisible by 3. Since, for all possible natural numbers x, $x^2 + 3\alpha + 1$ is not divisible by 3. Thus, $v_3(x^2 + 3\alpha + 1) = 0$ for all natural numbers x.

Future Work

Unfortunately, we have not yet proven that $v_3(x^2 + 3\beta + 1)$ is unbounded for all x or that $v_3(x^2 + 3\gamma + 0) = 0$ for $x = 3t$ or $v_3(x^2 + 3\gamma + 0) = 1$ for $x \neq 3t$. We hope to prove these conjectures and completely characterize $v_3(x^2 + a)$ for all natural numbers a.

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References

1. Boultinghouse, W.; Long, J.; Kozhushkina, O.; Trulen, J.; 2-adic Valuations of Quadratic Sequences, Journal of Integer Sequences, 24 (2021), 22 Available at https://cs.uwaterloo.ca/journals/JIS/VOL24/Trulen/trulen5.html

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